

Abelian integrals and limit cycles for a class of cubic polynomial vector fields of Lotka-Volterra type with a rational first integral of degree 2

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Abstract In this paper, we study the number of limit cycles which bifurcate from the periodic orbits of cubic polynomial vector fields of Lotka-Volterra type having a rational first integral of degree 2, under polynomial perturbations of degree n . The analysis is carried out by estimating the number of zeros of the corresponding Abelian integrals. Moreover, using *Chebyshev criterion*, we show that the sharp upper bound for the number of zeros of the Abelian integrals defined on each period annulus is 3 for $n = 3$. The simultaneous bifurcation and distribution of limit cycles for the system with two period annuli under cubic polynomial perturbations are considered. All configurations (u, v) with $0 \leq u, v \leq 3, u + v \leq 5$ are realizable.

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1 Introduction and statement of the main results

As is known, in the study of the qualitative theory of real planar differential systems, one of the important open problems is the determination of limit cycles. The second part of the famous Hilbert's 16th problem, proposed in 1900, asks for an upper bound on Hilbert number $H(n)$ and position of limit cycles for all planar polynomial differential systems of degree n , but it is still open even for $n = 2$. It is so difficult that the weak form of this problem has been introduced. A classical way to obtain limit cycles is that perturbing the periodic orbits of a center. Let us consider the planar polynomial vector fields $X_\varepsilon = X_0 + \varepsilon Y$, where $0 < \varepsilon \ll 1$ and

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$X_0 = (-H_y/R, H_x/R)$ is a polynomial vector field having a continuum of periodic orbits. H is a first integral of X_0 and R is an integrating factor. If $R = 1$, we call X_0 a Hamiltonian vector field, otherwise, we say that it's a non-Hamiltonian integrable vector field. In order to study the periodic orbits of X_ε that remain among all the periodic orbits of X_0 , it is necessary to study the number of zeros of an (generalized) Abelian integral, also known as the first order Melnikov function, i.e.,

$$M(h) = \oint_{H=h} R(Y_1(x, y)dy - Y_2(x, y)dx), \quad (1.1)$$

where $\{H = h, h \in (h_0, h_1)\}$ are periodic orbits of X_0 , and $Y_i(x, y), i = 1, 2$, are the two components of Y .

To the best of our knowledge, many authors have investigated the limit cycles for the quadratic Hamiltonian systems and non-Hamiltonian integrable systems under polynomial perturbations (e.g., [4, 6, 7, 10, 11, 12, 13, 15, 19] and references therein). However, the studies on cubic and higher degree systems are relatively few (e.g., [1, 2, 12, 14, 16, 18, 20]).

In this paper, we concern with the number of zeros of Abelian integrals for a class of cubic non-Hamiltonian integrable systems of Lotka-Volterra type with a rational first integral of degree 2, the integrating factor of which consists of x and y . In general, it is difficult to study the bifurcation of limit cycles for a non-Hamiltonian integrable systems with integrating factors including both the variables x and y , thus the technique and the results are few (e.g., [1, 2, 14]).

The authors [3] first studied the planar cubic polynomial vector fields of Lotka-Volterra type with a rational first integral of degree 2 and got 28 non-topologically equivalent phase portraits, among which there are only 2 cases having at least one center in the finite plane, as shown in Figure 1 (see Figure 1 of [3]).

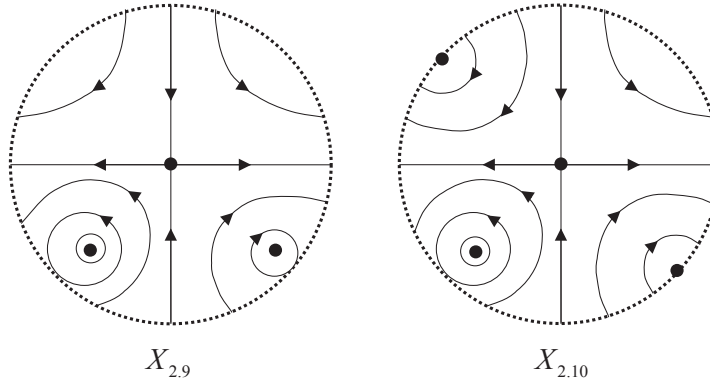


Figure 1: All the non-topologically equivalent phase portraits with at least one center of a planar cubic polynomial vector field of Lotka-Volterra type with a rational first integral of degree 2.

The system with a center in [3] is a differential system of the form

$$\begin{aligned}\dot{x} &= x(1 + bx + x^2 - y^2), \\ \dot{y} &= y(-1 - cy + x^2 - y^2),\end{aligned}\tag{1.2}$$

which has a first integral

$$H = \frac{1 + bx + cy + x^2 + y^2}{xy}$$

with integrating factor $x^{-2}y^{-2}$.

When $0 \leq b < 2$, $0 \leq c < 2$ and $b \neq c$, the phase portrait of system (1.2) corresponds to $X_{2,9}$ in Figure 1; when $0 < b = c < 2$, the phase portrait of system (1.2) is $X_{2,10}$ in Figure 1. For convenience, we will also denote the two subclasses of system (1.2) by $X_{2,9}$ and $X_{2,10}$, respectively.

System $X_{2,9}$ has three finite singular points: a hyperbolic saddle at $O(0,0)$ and two centers at $C_{\pm}(\delta(-b\delta \pm c\gamma)/2(\delta^2 - \gamma^2), \gamma(-c\gamma \pm b\delta)/2(\gamma^2 - \delta^2))$, where $\gamma = \sqrt{4 - b^2}$, $\delta = \sqrt{4 - c^2}$. C_+ is located in the third quadrant while C_- is located in the fourth (resp. second) quadrant if $b < c$ (resp. $b > c$). If $b > c$, taking a change $(x, y, t, b, c) \rightarrow (y, x, -t, c, b)$, system $X_{2,9}$ is reduced to that in the case $b < c$. Thus without loss of generality, it suffices to consider the latter case. $H = h_{\pm} = (-bc \pm \gamma\delta)/2$ correspond to C_{\pm} . There are two families of periodic orbits $\Gamma_h : H = h$, $h \in (-2, h_-) \cup (h_+, 2)$, which surround the centers C_- and C_+ , respectively.

System $X_{2,10}$ has two finite singular points. $O(0,0)$ is a hyperbolic saddle and $C(-1/b, -1/b)$ is a center, which is located in the third quadrant. There is a family of periodic orbits $\Gamma_h : H = h$, $h \in (2 - b^2, 2)$, which surrounds the center C . $H = 2 - b^2$ is the value of the center.

In what follows we are going to study the polynomial perturbations of these two subclasses of system (1.2):

$$\begin{aligned}\dot{x} &= x(1 + bx + x^2 - y^2) + \varepsilon f(x, y), \\ \dot{y} &= y(-1 - cy + x^2 - y^2) + \varepsilon g(x, y),\end{aligned}\tag{1.3}$$

where $f(x, y)$ and $g(x, y)$ are polynomials in the variables x and y with $\max\{\deg f, \deg g\} = n$.

Consider (1.3) with $0 < \varepsilon \ll 1$. Let $H_{X_{2,9}}(n)$ and $H_{X_{2,10}}(n)$ be the maximum number of zeros (taking into account their multiplicity) of the Abelian integrals associated with the two systems:

$$\begin{aligned}X_{2,9}, \quad & \text{i.e., } 0 \leq b < c < 2, \\ X_{2,10}, \quad & \text{i.e., } 0 < b = c < 2,\end{aligned}$$

respectively. Note that system $X_{2,9}$ has two period annuli and here $H_{X_{2,9}}(n)$ denotes the maximum number of zeros of Abelian integral $M(h)$ on the interval $(-2, h-)$ or $(h+, 2)$.

The main results of this paper are:

Theorem 1. $H_{X_{2,9}}(n) \leq 1$ if $n \leq 2$ and $H_{X_{2,9}}(n) \leq 2n - 3$ if $n \geq 3$.

Theorem 2. $H_{X_{2,10}}(n) \leq 1$ if $n \leq 2$ and $H_{X_{2,10}}(n) \leq [(3n - 3)/2]$ if $n \geq 3$.

Remark 3. By Proposition 1 of [14], it is easy to verify that the systems $X_{2,9}$ and $X_{2,10}$ can be transformed into S^* :

$$\begin{aligned}\dot{x} &= -y + \beta x^2 - 2\alpha xy - \beta y^2 + x^2 y, \\ \dot{y} &= x + \alpha x^2 + 2\beta xy - \alpha y^2 + xy^2,\end{aligned}$$

in Table I of [14]. Thus the centers of these two systems are isochronous. Moreover, it follows from the first integral that system $X_{2,10}$ is reversible (with respect to the straight line $y = x$) while system $X_{2,9}$ is not. In fact, by the affine transformation

$$(x, y, t) \rightarrow \left(y - x, \frac{b(x + y) + 2}{\sqrt{4 - b^2}}, -\frac{\sqrt{4 - b^2}}{b} t \right),$$

system $X_{2,10}$ is transformed into S^* with $\alpha = 0$, $\beta = -\sqrt{4 - b^2}/2$, which belongs to the reversible case (A) of [14]. As a special one, our result is better than that of the general case (see Theorem 2 of [14]). Similarly, by the affine transformation

$$(x, y, t) \rightarrow \left(\frac{x + y - \xi - \eta}{\sqrt{1 - (\xi + \eta)^2}}, \frac{y - x + \xi - \eta}{\sqrt{(\xi - \eta)^2 - 1}}, -\sqrt{4\xi^2\eta^2 - (\xi^2 + \eta^2 - 1)^2} t \right),$$

system $X_{2,9}$ is reduced to S^* with $\alpha = (\xi + \eta)\sqrt{1 - (\xi + \eta)^2}/(4\xi\eta)$, $\beta = (\xi - \eta)\sqrt{(\xi - \eta)^2 - 1}/(4\xi\eta)$, where $\xi = -\delta\zeta$, $\eta = \gamma\zeta$, $\zeta = (b\delta + c\gamma)/2(\delta^2 - \gamma^2)$. Since $\alpha\beta \neq 0$, system $X_{2,9}$ is not reversible. The problem that how many limit cycles bifurcate from the period annuli of system $X_{2,9}$ has not yet been studied.

Different from the classical methods which are used to study the number of limit cycles that bifurcate from the periodic orbits of a center, for example Poincaré-Melnikov integral method, Picard-Fuchs equation method [10, 20], inverse integrating factor method [8], averaging method [2] and so on, this paper takes advantage of some symmetric properties of the first integral and integrating factor to give the number of zeros of the Abelian integrals by direct computation. Compared with the method using Picard-Fuchs equation, our method gives the better upper bound. The computational approach has similarities with the technique used in [13, 14], but we do not use Green's theorem which makes the work harder than computing the usual single Abelian integral directly in this paper.

Recently, Llibre *et al.* also study limit cycles of cubic polynomial differential systems with rational first integrals of degree 2 under polynomial perturbations of degree 3 in [16] using averaging method. They give six families of cubic polynomial differential systems, denote by P_k for $k = 1, 2, \dots, 6$, where P_3 and P_5 are equivalent to ours. However, they only give the exact upper bound for P_1, P_2, P_4 and P_6 , and give an example of class P_5 that has at most 3 limit cycles.

We investigate the sharp upper bound for the number of zeros of the Abelian integrals with respect to the two systems for $n = 3$. The result is the following one.

Theorem 4. $H_{X_{2.9}}(3) = 3$ and $H_{X_{2.10}}(3) = 3$.

As is introduced above, there are two families of periodic orbits of system $X_{2.9}$. It is natural to consider the simultaneous bifurcation and distribution of limit cycles that emerge from both period annuli (e.g., [5, 7]). The configuration of limit cycles (u, v) , $u \geq 0, v \geq 0$, is considered to be achievable if, for ε small enough, exactly u (resp. v) limit cycles bifurcate from the periodic orbits surrounding C_+ (resp. C_-). Though, up to first order in ε , three limit cycles can emerge from each period annulus of system $X_{2.9}$ under cubic polynomial perturbations, it does not mean that the configuration (3,3) can be achievable (see for instance [6, 15]). In the present paper, we give a positive answer, i.e., (3,3) is impossible.

Theorem 5. *Under cubic polynomial perturbations, all configurations (u, v) of limit cycles bifurcated from the two period annuli of system $X_{2.9}$ can be realized, where $0 \leq u, v \leq 3, u + v \leq 5$.*

The paper is organized as follows: in sections 2 and 3, we study the number of zeros of the Abelian integrals for systems $X_{2.9}$ and $X_{2.10}$, respectively. Upper bounds for $H_{X_{2.9}}(n)$ and $H_{X_{2.10}}(n)$ are obtained. Section 4 focuses on the analysis of the least upper bound for the number of zeros of the Abelian integrals associated to the two systems for $n = 3$. *Chebyshev criterion* is used to determine $H_{X_{2.9}}(3)$ and $H_{X_{2.10}}(3)$. Finally, we give the simultaneous bifurcation and distribution of limit cycles bifurcated from the two period annuli of system $X_{2.9}$ under cubic polynomial perturbations in section 5.

2 Zeros of the Abelian integral for system $X_{2.9}$

In this section, we will study $H_{X_{2.9}}(n)$. Let Γ_h be the closed component of the algebraic curve

$$H(x, y, b, c) = \frac{1 + bx + cy + x^2 + y^2}{xy} = h$$

and Γ'_h be the closed component of the algebraic curve

$$H(x, y, c, b) = \frac{1 + cx + by + x^2 + y^2}{xy} = h.$$

The Abelian integral associated to system $X_{2,9}$ is defined as

$$\begin{aligned} M(h) &= \oint_{\Gamma_h} x^{-2}y^{-2}(f dy - g dx) \\ &= \sum_{i+j=0}^n \oint_{\Gamma_h} (a_{ij}x^{i-2}y^{j-2}dy - b_{ij}x^{i-2}y^{j-2}dx), \quad h \in (-2, h_-) \cup (h_+, 2), \end{aligned} \quad (2.1)$$

where Γ_h has the positive (resp. negative) orientation if it surrounds C_+ (resp. C_-).

Denote

$$I_{ij}(h) = \oint_{\Gamma_h} x^{i-2}y^{j-2}dx, \quad \bar{I}_{ij}(h) = \oint_{\Gamma'_h} x^{i-2}y^{j-2}dx. \quad (2.2)$$

Proposition 6. *$M(h)$ has the following expression:*

$$M(h) = \sum_{i+j=0}^n (-a_{ji}\bar{I}_{ij}(h) - b_{ij}I_{ij}(h)), \quad h \in (-2, h_-) \cup (h_+, 2). \quad (2.3)$$

Proof. Suppose the level curve Γ_h is a periodic orbit that surrounds the center C_+ of system $X_{2,9}$, then

$$\begin{aligned} &\oint_{\Gamma_h} x^{i-2}y^{j-2}dy \\ &= \int_{y_1}^{y_2} \left(\frac{hy - b}{2} + \sqrt{\Psi(y)} \right)^{i-2} y^{j-2}dy + \int_{y_2}^{y_1} \left(\frac{hy - b}{2} - \sqrt{\Psi(y)} \right)^{i-2} y^{j-2}dy, \end{aligned}$$

where $\Psi(y) = (h^2/4 - 1)y^2 - (bh/2 + c)y + b^2/4 - 1$ and y_1, y_2 denote the two roots of the equation $\Psi(y) = 0$ with $y_1 < y_2$. Set $y = x$ in the right-hand-side of the last equality, then

$$\begin{aligned} &\oint_{\Gamma_h} x^{i-2}y^{j-2}dy \\ &= \int_{y_1}^{y_2} \left(\frac{hx - b}{2} + \sqrt{\Psi(x)} \right)^{i-2} x^{j-2}dx + \int_{y_2}^{y_1} \left(\frac{hx - b}{2} - \sqrt{\Psi(x)} \right)^{i-2} x^{j-2}dx \quad (2.4) \\ &= - \oint_{\Gamma'_h} y^{i-2}x^{j-2}dx = -\bar{I}_{ji}(h), \end{aligned}$$

where Γ'_h has the same orientation as Γ_h . The second equality holds since y_1, y_2 are also the roots of the equation $\Psi(x) = 0$.

The proposition follows from (2.1), (2.2) and (2.4). \square

By Proposition 6, owing to the fact that the form and algorithm of the two parts of $M(h)$ are similar, we only need to calculate

$$\sum_{i+j=0}^n b_{ij}I_{ij}(h) = M_1(h) + M_2(h) + M_3(h), \quad h \in (-2, h_-) \cup (h_+, 2), \quad (2.5)$$

where

$$\begin{aligned} M_1(h) &= \sum_{\substack{j \geq 2, \\ i+j \leq n}} b_{ij} I_{ij}(h) = \sum_{\substack{j \geq 2, \\ i+j \leq n}} b_{ij} \oint_{\Gamma_h} x^{i-2} y^{j-2} dx, \\ M_2(h) &= \sum_{i=0}^{n-1} b_{i1} I_{i1}(h) = \sum_{i=0}^{n-1} b_{i1} \oint_{\Gamma_h} x^{i-2} y^{-1} dx, \\ M_3(h) &= \sum_{i=0}^n b_{i0} I_{i0}(h) = \sum_{i=0}^n b_{i0} \oint_{\Gamma_h} x^{i-2} y^{-2} dx. \end{aligned}$$

Let $\Delta = (h^2/4 - 1)x^2 - (ch/2 + b)x + c^2/4 - 1$ and x_1, x_2 are the two roots of the equation $\Delta = 0$ with $x_1 < x_2$. Then

$$x_1 + x_2 = -\frac{2(2b + ch)}{4 - h^2}, \quad x_1 x_2 = \frac{4 - c^2}{4 - h^2}. \quad (2.6)$$

We have that

$$I_{ij}(h) = \pm \int_{x_1}^{x_2} x^{i-2} \left[\left(\frac{hx - c}{2} - \sqrt{\Delta} \right)^{j-2} - \left(\frac{hx - c}{2} + \sqrt{\Delta} \right)^{j-2} \right] dx. \quad (2.7)$$

Remark 7. The appearance of “ \pm ” in (2.7) comes from the fact that Γ_h has the positive orientation when $h \in (h_+, 2)$ and the negative orientation when $h \in (-2, h_-)$. However, it is easy to verify that the sign “ \pm ” does not affect the maximum number of zeros of the Abelian integral defined on each interval. Thus we will drop it in the computation of $M_i(h)$, $i = 1, 2, 3$ for $h \in (-2, h_-) \cup (h_+, 2)$. The results obtained differ from the original functions at most by a negative sign.

First, we get that

$$\begin{aligned} M_1(h) &= \sum_{\substack{j \geq 2, \\ i+j \leq n}} b_{ij} \int_{x_1}^{x_2} x^{i-2} \left[\left(\frac{hx - c}{2} - \sqrt{\Delta} \right)^{j-2} - \left(\frac{hx - c}{2} + \sqrt{\Delta} \right)^{j-2} \right] dx \\ &= \sum_{\substack{j \geq 3, \\ i+j \leq n}} b_{ij} \int_{x_1}^{x_2} x^{i-2} \left(\sum_{k=0}^{j-3} \tilde{m}_{j,k}(h) x^k \sqrt{\Delta} \right) dx \\ &= \sum_{k=0}^{n-3} m_k(h) J_k(h), \end{aligned} \quad (2.8)$$

where $\tilde{m}_{j,k}(h), m_k(h)$ are polynomials of h with $\deg \tilde{m}_{j,k}(h) \leq k, \deg m_k(h) \leq k$, and

$$J_k(h) = \int_{x_1}^{x_2} x^{k-2} \sqrt{\Delta} dx = \sqrt{1 - h^2/4} \int_{x_1}^{x_2} x^{k-2} \sqrt{(x_2 - x)(x - x_1)} dx.$$

In order to study $J_k(h)$, we use two different transformations. If $\sqrt{(x_2 - x)(x - x_1)} = t(x - x_1)$, then

$$J_k(h) = \sqrt{4 - h^2} (x_2 - x_1)^2 \int_0^\infty \frac{t^2 (x_2 + t^2 x_1)^{k-2}}{(1 + t^2)^{k+1}} dt,$$

and if $\sqrt{(x_2 - x)(x - x_1)} = t(x_2 - x)$, then

$$J_k(h) = \sqrt{4 - h^2} (x_2 - x_1)^2 \int_0^\infty \frac{t^2 (x_1 + t^2 x_2)^{k-2}}{(1 + t^2)^{k+1}} dt.$$

Thus, by (2.6), for $k \geq 2$,

$$\begin{aligned} J_k(h) &= \sqrt{4 - h^2} (x_2 - x_1)^2 \int_0^\infty \frac{t^2 [(x_2 + t^2 x_1)^{k-2} + (x_1 + t^2 x_2)^{k-2}]}{2(1 + t^2)^{k+1}} dt \\ &= \sqrt{4 - h^2} (x_2 - x_1)^2 \sum_{i+2j=k-2} d_{ij} (x_1 + x_2)^i (x_1 x_2)^j \\ &= \frac{16(h^2 + bch + b^2 + c^2 - 4)}{(4 - h^2)^{3/2}} \sum_{i+2j=k-2} d_{ij} \left[-\frac{2(2b + ch)}{4 - h^2} \right]^i \left(\frac{4 - c^2}{4 - h^2} \right)^j \\ &= (4 - h^2)^{-(k-1/2)} P_k(h), \end{aligned} \tag{2.9}$$

where $d_{ij}, i + 2j = k - 2$ are constants and $P_k(h)$ denotes a polynomial of degree k . By direct computations, we get that

$$\begin{aligned} J_1(h) &= \int_{x_1}^{x_2} x^{-1} \sqrt{\Delta} dx = \begin{cases} -\frac{2b + ch}{2\sqrt{4 - h^2}} \pi - \frac{\sqrt{4 - c^2}}{2} \pi, & h \in (-2, h_-), \\ -\frac{2b + ch}{2\sqrt{4 - h^2}} \pi + \frac{\sqrt{4 - c^2}}{2} \pi, & h \in (h_+, 2), \end{cases} \\ J_0(h) &= \int_{x_1}^{x_2} x^{-2} \sqrt{\Delta} dx = \begin{cases} -\frac{2b + ch}{2\sqrt{4 - c^2}} \pi - \frac{\sqrt{4 - h^2}}{2} \pi, & h \in (-2, h_-), \\ \frac{2b + ch}{2\sqrt{4 - c^2}} \pi - \frac{\sqrt{4 - h^2}}{2} \pi, & h \in (h_+, 2). \end{cases} \end{aligned} \tag{2.10}$$

It follows from (2.8), (2.9) and (2.10) that

$$\begin{aligned} M_1(h) &= m_0(h) J_0(h) + m_1(h) J_1(h) + \sum_{k=2}^{n-3} m_k(h) J_k(h) \\ &= \hat{P}_1(h) + \frac{\hat{P}_{2n-6}(h)}{(4 - h^2)^{n-7/2}}, \end{aligned} \tag{2.11}$$

where $\hat{P}_1(h)$ and $\hat{P}_{2n-6}(h)$ are polynomials with $\deg \hat{P}_1(h) \leq 1$ and $\deg \hat{P}_{2n-6}(h) \leq 2n - 6$, respectively.

Next we calculate $M_2(h)$ and $M_3(h)$.

$$\begin{aligned}
 M_2(h) &= \sum_{i=0}^{n-1} b_{i1} \int_{x_1}^{x_2} x^{i-2} \left[\frac{1}{(hx-c)/2 - \sqrt{\Delta}} - \frac{1}{(hx-c)/2 + \sqrt{\Delta}} \right] dx \\
 &= \sum_{i=0}^{n-1} b_{i1} \int_{x_1}^{x_2} \frac{2x^{i-2}\sqrt{\Delta}}{x^2 + bx + 1} dx \\
 &= \sum_{i=0}^3 \tilde{b}_{i1} S_i(h) + \sum_{i=4}^{n-1} \tilde{b}_{i1} \int_{x_1}^{x_2} x^{i-4} \sqrt{\Delta} dx \\
 &= \sum_{i=0}^3 \tilde{b}_{i1} S_i(h) + \sum_{i=4}^{n-1} \tilde{b}_{i1} J_{i-2}(h),
 \end{aligned} \tag{2.12}$$

where $\tilde{b}_{i1}, i = 0, 1, \dots, n-1$, are linear combinations of $b_{i1}, i = 0, 1, \dots, n-1$, and

$$S_i(h) = \int_{x_1}^{x_2} \frac{2x^{i-2}\sqrt{\Delta}}{x^2 + bx + 1} dx, \quad i = 0, 1, 2, 3.$$

If $h \in (-2, h_-)$, then

$$\begin{aligned}
 S_0(h) &= \frac{(b^2-2)c + bh}{\sqrt{4-b^2}}\pi - \frac{(c^2-2)b + ch}{\sqrt{4-c^2}}\pi, \\
 S_1(h) &= -\frac{bc+2h}{\sqrt{4-b^2}}\pi - \sqrt{4-c^2}\pi, \\
 S_2(h) &= \frac{2c+bh}{\sqrt{4-b^2}}\pi - \sqrt{4-h^2}\pi, \\
 S_3(h) &= -\frac{bc+(b^2-2)h}{\sqrt{4-b^2}}\pi - \frac{-2b+ch+bh^2}{\sqrt{4-h^2}}\pi,
 \end{aligned} \tag{2.13}$$

and if $h \in (h_+, 2)$, then

$$\begin{aligned}
 S_0(h) &= \frac{(b^2-2)c + bh}{\sqrt{4-b^2}}\pi + \frac{(c^2-2)b + ch}{\sqrt{4-c^2}}\pi, \\
 S_1(h) &= -\frac{bc+2h}{\sqrt{4-b^2}}\pi + \sqrt{4-c^2}\pi,
 \end{aligned} \tag{2.14}$$

$S_2(h)$ and $S_3(h)$ are the same as (2.13). By (2.9), (2.12), (2.13) and (2.14), we obtain

$$M_2(h) = \tilde{P}_1(h) + \frac{\tilde{P}_{2n-6}(h)}{(4-h^2)^{n-7/2}}, \tag{2.15}$$

where $\tilde{P}_1(h)$ and $\tilde{P}_{2n-6}(h)$ are polynomials with $\deg \tilde{P}_1(h) \leq 1$ and $\deg \tilde{P}_{2n-6}(h) \leq 2n-6$, respectively.

The technique used in the computation of $M_3(h)$ is shown as follows:

$$\begin{aligned}
M_3(h) &= \sum_{i=0}^n b_{i0} \int_{x_1}^{x_2} x^{i-2} \left[\frac{1}{((hx-c)/2 - \sqrt{\Delta})^2} - \frac{1}{((hx-c)/2 + \sqrt{\Delta})^2} \right] dx \\
&= \sum_{i=0}^n b_{i0} \int_{x_1}^{x_2} \frac{2x^{i-2}(hx-c)\sqrt{\Delta}}{(x^2+bx+1)^2} dx \\
&= \sum_{i=0}^3 b_{i0} R_i(h) + \sum_{i=4}^n b_{i0} \int_{x_1}^{x_2} \frac{2x^{i-2}(hx-c)\sqrt{\Delta}}{(x^2+bx+1)^2} dx \\
&= \mu(h)S_2(h) + \nu(h)S_3(h) + \sum_{i=0}^3 \bar{b}_{i0} R_i(h) + \sum_{i=5}^n \bar{b}_{i0} \omega_i(h) J_{i-3}(h),
\end{aligned} \tag{2.16}$$

where $\bar{b}_{i0}, i = 0, 1, 2, 3, 5, \dots, n$, are linear combinations of $b_{i0}, i = 0, 1, \dots, n$. $\mu(h)$, $\nu(h)$ and $\omega_i(h)$ are polynomials of degree at most 1, and

$$R_i(h) = \int_{x_1}^{x_2} \frac{2x^{i-2}(hx-c)\sqrt{\Delta}}{(x^2+bx+1)^2} dx, \quad i = 0, 1, 2, 3.$$

Further calculations show that when $h \in (-2, h_-)$,

$$\begin{aligned}
R_0(h) &= \frac{2(-8 + 6c^2 - 6b^2(-1 + c^2) + b^4(-1 + c^2))\pi}{(4 - b^2)^{3/2}} + \frac{2bc(-3 + c^2)\pi}{\sqrt{4 - c^2}} \\
&\quad + h \left(\frac{2b(-6 + b^2)c\pi}{(4 - b^2)^{3/2}} + \frac{2(-2 + c^2)\pi}{\sqrt{4 - c^2}} \right) - \frac{4h^2\pi}{(4 - b^2)^{3/2}}, \\
R_1(h) &= \frac{2bc^2\pi}{(4 - b^2)^{3/2}} + \frac{b(c^2 - 2)\pi}{\sqrt{4 - b^2}} + c\sqrt{4 - c^2}\pi + \frac{8ch\pi}{(4 - b^2)^{3/2}} + \frac{2bh^2\pi}{(4 - b^2)^{3/2}}, \\
R_2(h) &= \frac{-4(-4 + b^2 + c^2 + bch + h^2)\pi}{(4 - b^2)^{3/2}}, \\
R_3(h) &= \frac{2b(-4 + b^2 + c^2)\pi}{(4 - b^2)^{3/2}} + \frac{8ch\pi}{(4 - b^2)^{3/2}} - \frac{b(-6 + b^2)h^2\pi}{(4 - b^2)^{3/2}} - h\sqrt{4 - h^2}\pi,
\end{aligned} \tag{2.17}$$

and when $h \in (h_+, 2)$,

$$\begin{aligned}
R_0(h) &= \frac{2(-8 + 6c^2 - 6b^2(-1 + c^2) + b^4(-1 + c^2))\pi}{(4 - b^2)^{3/2}} - \frac{2bc(-3 + c^2)\pi}{\sqrt{4 - c^2}} \\
&\quad + h \left(\frac{2b(-6 + b^2)c\pi}{(4 - b^2)^{3/2}} - \frac{2(-2 + c^2)\pi}{\sqrt{4 - c^2}} \right) - \frac{4h^2\pi}{(4 - b^2)^{3/2}}, \\
R_1(h) &= \frac{2bc^2\pi}{(4 - b^2)^{3/2}} + \frac{b(c^2 - 2)\pi}{\sqrt{4 - b^2}} - c\sqrt{4 - c^2}\pi + \frac{8ch\pi}{(4 - b^2)^{3/2}} + \frac{2bh^2\pi}{(4 - b^2)^{3/2}},
\end{aligned} \tag{2.18}$$

$R_2(h)$ and $R_3(h)$ are the same as (2.17). Hence by (2.9), (2.13), (2.14), (2.16), (2.17) and (2.18), we get

$$M_3(h) = \bar{P}_2(h) + \frac{\bar{P}_{2n-5}(h)}{(4 - h^2)^{n-7/2}}, \tag{2.19}$$

where $\bar{P}_2(h)$ and $\bar{P}_{2n-5}(h)$ stand for polynomials of h with $\deg \bar{P}_2(h) \leq 2$ and $\deg \bar{P}_{2n-5}(h) \leq 2n-5$, respectively.

Proof of Theorem 1. From (2.5), (2.11), (2.15), (2.19) and Proposition 6, together with the similar form and algorithm of the two parts of $M(h)$, we get the Abelian integral

$$M(h) = \begin{cases} Q_2(h) + \frac{Q_{2n-5}(h)}{(4-h^2)^{n-7/2}} & \text{if } n \geq 3, \\ \tilde{Q}_2(h) & \text{if } n \leq 2, \end{cases} \quad (2.20)$$

where $Q_2(h)$, $\tilde{Q}_2(h)$ and $Q_{2n-5}(h)$ are polynomials with $\deg Q_2(h) \leq 2$, $\deg \tilde{Q}_2(h) \leq 2$ and $\deg Q_{2n-5}(h) \leq 2n-5$, respectively. We remark that $M(h)$ has the same form as (2.20) on $(-2, h_-)$ and $(h_+, 2)$, respectively. But on these two intervals, they are different functions (it is clear to see from (2.10), (2.13), (2.14) and so on).

Obviously, $M(h)$ is a polynomial of degree at most 2 if $n \leq 2$ and it has at most 2 zeros. Moreover, it vanishes at the value of the center. Thus it has at most one zero in each open interval.

$M(h)$ does not have a rational form if $n \geq 3$. But we can get the estimation of the number of zeros of $M(h)$ by taking derivatives three times. Therefore

$$\#\{h | M^{(3)}(h) = 0\} \leq 2n-5,$$

where $\#$ denotes the number of elements of a finite set. On the other hand, note that $M(h_{\pm}) = 0$. Hence

$$\begin{aligned} & \max\{\#\{h \in (-2, h_-) | M(h) = 0\}, \#\{h \in (h_+, 2) | M(h) = 0\}\} \\ & \leq 2n-5 + 3 - 1 = 2n-3, \end{aligned}$$

i.e.,

$$H_{X_{2.9}}(n) \leq 2n-3.$$

3 Zeros of the Abelian integral for system $X_{2.10}$

This section is devoted to investigate the polynomial perturbations of system $X_{2.10}$, i.e., system (1.3) with $b = c$.

Firstly, we have the following result by the same argument as the proof of Proposition 6.

Proposition 8. *The Abelian integral associated to the system $X_{2,10}$ is*

$$\begin{aligned}
 M(h) &= \oint_{\Gamma_h} x^{-2}y^{-2}(f dy - g dx) \\
 &= \sum_{i+j=0}^n \oint_{\Gamma_h} (a_{ij}x^{i-2}y^{j-2}dy - b_{ij}x^{i-2}y^{j-2}dx) \\
 &= \sum_{i+j=0}^n (-a_{ji} - b_{ij})I_{ij}(h), \quad h \in (2 - b^2, 2),
 \end{aligned} \tag{3.1}$$

where Γ_h is the closed component of the algebraic curve $1 + bx + by + x^2 + y^2 = hxy$ with the positive orientation, and

$$I_{ij}(h) = \oint_{\Gamma_h} x^{i-2}y^{j-2}dx.$$

Because the algorithm of $M(h)$ is the same as that in Section 2, we only give the main results.

$$M(h) = \sum_{i+j=0}^n c_{ij}I_{ij}(h) = M_1(h) + M_2(h) + M_3(h), \quad h \in (2 - b^2, 2), \tag{3.2}$$

where $c_{ij} = -a_{ji} - b_{ij}$, and

$$\begin{aligned}
 M_1(h) &= \sum_{\substack{j \geq 2, \\ i+j \leq n}} c_{ij}I_{ij}(h) = \sum_{\substack{j \geq 2, \\ i+j \leq n}} c_{ij} \oint_{\Gamma_h} x^{i-2}y^{j-2}dx, \\
 M_2(h) &= \sum_{i=0}^{n-1} c_{i1}I_{i1}(h) = \sum_{i=0}^{n-1} c_{i1} \oint_{\Gamma_h} x^{i-2}y^{-1}dx, \\
 M_3(h) &= \sum_{i=0}^n c_{i0}I_{i0}(h) = \sum_{i=0}^n c_{i0} \oint_{\Gamma_h} x^{i-2}y^{-2}dx.
 \end{aligned}$$

Denote $\Delta = (h^2/4 - 1)x^2 - (bh/2 + b)x + b^2/4 - 1$ and let x_1, x_2 be the two roots of the equation $\Delta = 0$ with $x_1 < x_2$. Then

$$x_1 + x_2 = -\frac{2b}{2-h}, \quad x_1x_2 = \frac{4-b^2}{4-h^2}. \tag{3.3}$$

Consequently,

$$M_1(h) = \sum_{k=0}^{n-3} m_k(h)J_k(h), \tag{3.4}$$

where $m_k(h)$ is a polynomial of h with $\deg m_k(h) \leq k$ and

$$J_k(h) = \int_{x_1}^{x_2} x^{k-2} \sqrt{\Delta} dx.$$

It is easy to verify that if $k \geq 2$,

$$\begin{aligned}
 J_k(h) &= \sqrt{4-h^2} (x_2 - x_1)^2 \sum_{i+2j=k-2} \bar{d}_{ij} (x_1 + x_2)^i (x_1 x_2)^j \\
 &= \frac{16(h - (2 - b^2))}{(2 - h)\sqrt{4 - h^2}} \sum_{i+2j=k-2} \bar{d}_{ij} \left(-\frac{2b}{2-h}\right)^i \left(\frac{4-b^2}{4-h^2}\right)^j \\
 &= \frac{16(h - (2 - b^2))}{\sqrt{4 - h^2}(2 - h)^{k-1}(2 + h)^{[(k-2)/2]}} P_{[(k-2)/2]}(h),
 \end{aligned} \tag{3.5}$$

where $\bar{d}_{ij}, i+2j = k-2$ are constants, $[(k-2)/2]$ denotes the integer part of $(k-2)/2$ and $P_{[(k-2)/2]}(h)$ is a polynomial of degree $[(k-2)/2]$. A direct computation leads to

$$\begin{aligned}
 J_1(h) &= -\frac{b(2+h)}{2\sqrt{4-h^2}}\pi + \frac{\sqrt{4-b^2}}{2}\pi, \\
 J_0(h) &= \frac{b(2+h)}{2\sqrt{4-b^2}}\pi - \frac{\sqrt{4-h^2}}{2}\pi.
 \end{aligned} \tag{3.6}$$

It follows from (3.4), (3.5) and (3.6) that

$$M_1(h) = \hat{P}_1(h) + \frac{\hat{P}_{[(3n-9)/2]}(h)}{\sqrt{4-h^2}(2-h)^{n-4}(2+h)^{[(n-5)/2]}}, \tag{3.7}$$

where $\hat{P}_1(h)$ and $\hat{P}_{[(3n-9)/2]}(h)$ are polynomials with $\deg \hat{P}_1 \leq 1$ and $\deg \hat{P}_{[(3n-9)/2]} \leq [(3n-9)/2]$, respectively.

In addition,

$$\begin{aligned}
 M_2(h) &= \sum_{i=0}^{n-1} c_{i1} \int_{x_1}^{x_2} x^{i-2} \left(\frac{1}{(hx-b)/2 - \sqrt{\Delta}} - \frac{1}{(hx-b)/2 + \sqrt{\Delta}} \right) dx \\
 &= \sum_{i=0}^3 \tilde{c}_{i1} S_i(h) + \sum_{i=4}^{n-1} \tilde{c}_{i1} J_{i-2}(h),
 \end{aligned} \tag{3.8}$$

where $\tilde{c}_{i1}, i = 0, 1, \dots, n-1$ are linear combinations of $c_{i1}, i = 0, 1, \dots, n-1$, and

$$S_i(h) = \int_{x_1}^{x_2} \frac{2x^{i-2}\sqrt{\Delta}}{x^2 + bx + 1} dx, \quad i = 0, 1, 2, 3.$$

By direct calculation we obtain that

$$\begin{aligned}
 S_0(h) &= -bS_1(h) = \frac{2b(h - (2 - b^2))}{\sqrt{4 - b^2}}\pi, \\
 S_2(h) &= \frac{b(2+h)}{\sqrt{4-b^2}}\pi - \sqrt{4-h^2}\pi, \\
 S_3(h) &= -\frac{b^2 + (b^2 - 2)h}{\sqrt{4-b^2}}\pi - \frac{b(-2+h+h^2)}{\sqrt{4-h^2}}\pi.
 \end{aligned} \tag{3.9}$$

Therefore, we get from (3.5), (3.8) and (3.9) that

$$M_2(h) = \tilde{P}_1(h) + \frac{\tilde{P}_{[(3n-9)/2]}(h)}{\sqrt{4-h^2}(2-h)^{n-4}(2+h)^{[(n-5)/2]}}, \quad (3.10)$$

where $\tilde{P}_1(h)$ and $\tilde{P}_{[(3n-9)/2]}(h)$ are polynomials with $\deg \tilde{P}_1 \leq 1$ and $\deg \tilde{P}_{[(3n-9)/2]} \leq [(3n-9)/2]$, respectively.

Finally, we have that

$$\begin{aligned} M_3(h) &= \sum_{i=0}^n c_{i0} \int_{x_1}^{x_2} x^{i-2} \left(\frac{1}{((hx-b)/2 - \sqrt{\Delta})^2} - \frac{1}{((hx-b)/2 + \sqrt{\Delta})^2} \right) dx \\ &= \sum_{i=0}^n c_{i0} \int_{x_1}^{x_2} \frac{2x^{i-2}(hx-b)\sqrt{\Delta}}{(x^2+bx+1)^2} dx \\ &= \mu(h)S_2(h) + \nu(h)S_3(h) + \sum_{i=0}^3 \bar{c}_{i0}R_i(h) + \sum_{i=5}^n \bar{c}_{i0}\omega_i(h)J_{i-3}(h), \end{aligned} \quad (3.11)$$

where $\bar{c}_{i0}, i = 0, 1, 2, 3, 5, \dots, n$, are linear combinations of $c_{i0}, i = 0, 1, \dots, n$. $\mu(h)$, $\nu(h)$ and $\omega_i(h)$ are polynomials of degree at most 1, and

$$R_i(h) = \int_{x_1}^{x_2} \frac{2x^{i-2}(hx-b)\sqrt{\Delta}}{(x^2+bx+1)^2} dx, \quad i = 0, 1, 2, 3.$$

A simple calculation shows that

$$\begin{aligned} R_0(h) &= -\frac{4(h-(2-b^2))(h-b^4+5b^2-2)\pi}{(4-b^2)^{3/2}}, \\ R_1(h) &= \frac{2b(h-(2-b^2))(h-b^2+6)\pi}{(4-b^2)^{3/2}}, \\ R_2(h) &= -\frac{4(h-(2-b^2))(h+2)\pi}{(4-b^2)^{3/2}}, \\ R_3(h) &= -\frac{b(h+2)(b^2h-6h-2b^2+4)\pi}{(4-b^2)^{3/2}} - h\sqrt{4-h^2}\pi. \end{aligned} \quad (3.12)$$

From (3.5), (3.9), (3.11) and (3.12), we obtain

$$M_3(h) = \bar{P}_2(h) + \frac{\bar{P}_{[(3n-7)/2]}(h)}{\sqrt{4-h^2}(2-h)^{n-4}(2+h)^{[(n-5)/2]}}, \quad (3.13)$$

where $\bar{P}_2(h)$ and $\bar{P}_{[(3n-7)/2]}(h)$ are polynomials with $\deg \bar{P}_2 \leq 2$ and $\deg \bar{P}_{[(3n-7)/2]} \leq [(3n-7)/2]$, respectively.

Proof of Theorem 2. It follows from Proposition 8, (3.2), (3.7), (3.10) and (3.13) that the Abelian integral is

$$M(h) = \begin{cases} Q_2(h) + \frac{Q_{[(3n-7)/2]}(h)}{\sqrt{4-h^2}(2-h)^{n-4}(2+h)^{[(n-5)/2]}} & \text{if } n \geq 3, \\ \tilde{Q}_2(h) & \text{if } n \leq 2, \end{cases} \quad (3.14)$$

where $Q_2(h)$, $\tilde{Q}_2(h)$ and $Q_{[(3n-7)/2]}(h)$ denote polynomials with $\deg Q_2 \leq 2$, $\deg \tilde{Q}_2 \leq 2$ and $\deg Q_{[(3n-7)/2]} \leq [(3n-7)/2]$, respectively.

If $n \leq 2$, then the proof is similar to the proof of Theorem 1 and hence is omitted.

If $n \geq 3$, in order to get the estimation of the number of zeros of $M(h)$, we take derivatives of $M(h)$ three times. Noting that $M(2-b^2) = 0$, it follows that

$$H_{X_{2.10}}(n) \leq \left\lceil \frac{3n-7}{2} \right\rceil + 3 - 1 = \left\lceil \frac{3n-3}{2} \right\rceil.$$

The proof is finished.

4 $H_{X_{2.9}}(3) = 3$ and $H_{X_{2.10}}(3) = 3$

In this section, we will study the sharp upper bound for the number of zeros of the Abelian integrals with respect to the two systems for $n = 3$, i.e., the determination of $H_{X_{2.9}}(3)$ and $H_{X_{2.10}}(3)$. For convenience, we will denote

$$U^- = (-2, h_-), \quad U^+ = (h_+, 2).$$

The exact expression of Abelian integral associated to system $X_{2.9}$ is given as follows:

$$\begin{aligned} M^\pm(h) &= \sum_{i+j=0}^3 (-a_{ji}\bar{I}_{ij}(h) - b_{ij}I_{ij}(h)) \\ &= a_1^\pm + a_2^\pm h + a_3^\pm h^2 + a_4^\pm \sqrt{4-h^2} + a_5^\pm h\sqrt{4-h^2}, \quad h \in U^\pm, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned}
 a_2^\pm &= b_{00} \left(\pm \frac{2bc(6-b^2)\pi}{(4-b^2)^{3/2}} + \frac{2(-2+c^2)\pi}{\sqrt{4-c^2}} \right) + a_{00} \left(-\frac{2bc(-6+c^2)\pi}{(4-c^2)^{3/2}} \pm \frac{2(-2+b^2)\pi}{\sqrt{4-b^2}} \right) \\
 &\quad - (b_{01} + a_{10}) \left(\pm \frac{b\pi}{\sqrt{4-b^2}} + \frac{c\pi}{\sqrt{4-c^2}} \right) \pm \frac{2b_{11}\pi}{\sqrt{4-b^2}} + \frac{2a_{11}\pi}{\sqrt{4-c^2}} \pm \frac{b(-b_{21} + a_{30})\pi}{\sqrt{4-b^2}} \\
 &\quad \pm \frac{4c(-2b_{10} + b_{20}b - 2b_{30})\pi}{(4-b^2)^{3/2}} - \frac{4b(2a_{01} - a_{02}c + 2a_{03})\pi}{(4-c^2)^{3/2}} - \frac{c(a_{12} - b_{03})\pi}{\sqrt{4-c^2}}, \\
 a_3^\pm &= \pm \frac{4(b_{00} + b_{20})\pi}{(4-b^2)^{3/2}} + \frac{4(a_{00} + a_{02})\pi}{(4-c^2)^{3/2}} \mp \frac{2b_{10}b\pi}{(4-b^2)^{3/2}} - \frac{2a_{01}c\pi}{(4-c^2)^{3/2}} \\
 &\quad \pm \frac{b_{30}b(-6+b^2)\pi}{(4-b^2)^{3/2}} + \frac{a_{03}c(-6+c^2)\pi}{(4-c^2)^{3/2}}, \\
 a_4^\pm &= \pm (b_{21} - b_{03} + a_{12} - a_{30})\pi, \\
 a_5^\pm &= \pm (b_{30} + a_{03})\pi,
 \end{aligned}$$

and

$$a_1^\pm = m^\pm a_2^\pm + n^\pm a_3^\pm + p^\pm a_4^\pm + q^\pm a_5^\pm \quad (4.2)$$

with

$$m^\pm = -h_\pm, \quad n^\pm = -h_\pm^2, \quad p^\pm = -\sqrt{4-h_\pm^2}, \quad q^\pm = -h_\pm\sqrt{4-h_\pm^2}.$$

Since $0 \leq b < c < 2$,

$$\frac{\partial(a_2^\pm, a_3^\pm, a_4^\pm, a_5^\pm)}{\partial(a_{11}, a_{01}, a_{12}, a_{03})} = -\frac{4c\pi^4}{(4-c^2)^2} \neq 0,$$

which means that $a_2^\pm, a_3^\pm, a_4^\pm, a_5^\pm$ are independent.

Denote

$$f_0^\pm(h) = h + m^\pm, \quad f_1^\pm(h) = h^2 + n^\pm, \quad f_2^\pm(h) = \sqrt{4-h^2} + p^\pm, \quad f_3^\pm(h) = h\sqrt{4-h^2} + q^\pm.$$

It follows from (4.1) and (4.2) that

$$M^\pm(h) = a_2^\pm f_0^\pm(h) + a_3^\pm f_1^\pm(h) + a_4^\pm f_2^\pm(h) + a_5^\pm f_3^\pm(h). \quad (4.3)$$

By Theorem 1, we know that $H_{X_{2,9}}(3) \leq 3$. Now, we use *Chebyshev criterion* to show that $H_{X_{2,9}}(3) = 3$, i.e., there exist the parameters $a_i^\pm, i = 2, 3, 4, 5$, such that $M^\pm(h)$ has exactly three zeros on U^\pm , respectively. We introduce the following definitions (see for instance [17]).

Let f_0, f_1, \dots, f_{n-1} be analytic functions on an open interval L of \mathbb{R} . $(f_0, f_1, \dots, f_{n-1})$ is a *Chebyshev system* on L if any nontrivial linear combination

$$\lambda_0 f_0(x) + \lambda_1 f_1(x) + \dots + \lambda_{n-1} f_{n-1}(x)$$

has at most $n - 1$ isolated zeros on L .

An ordered set $(f_0, f_1, \dots, f_{n-1})$ is a *complete Chebyshev system* on L if $(f_0, f_1, \dots, f_{i-1})$ is a Chebyshev system on L for all $i = 1, 2, \dots, n$.

An ordered set $(f_0, f_1, \dots, f_{n-1})$ is an *extended complete Chebyshev system* (in short, ECT-system) on L if, for all $i = 1, 2, \dots, n$, any nontrivial linear combination

$$\lambda_0 f_0(x) + \lambda_1 f_1(x) + \dots + \lambda_{i-1} f_{i-1}(x) \quad (4.4)$$

has at most $i - 1$ isolated zeros on L counted with multiplicities.

Remark 9. If $(f_0, f_1, \dots, f_{n-1})$ is an ECT-system on L , then for each $i = 1, 2, \dots, n$, there exists a linear of combination (4.4) with exactly $i - 1$ simple zeros on L (see for instance Remark 3.7 in [9]).

Lemma 10. (see [17]) $(f_0, f_1, \dots, f_{n-1})$ is an ECT-system on L if, and only if, for each $i = 1, 2, \dots, n$,

$$\Omega_i(x) = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{i-1}(x) \\ f'_0(x) & f'_1(x) & \cdots & f'_{i-1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(i-1)}(x) & f_1^{(i-1)}(x) & \cdots & f_{i-1}^{(i-1)}(x) \end{vmatrix} \neq 0$$

for all $x \in L$.

Simple computations show that, if $h \in U^\pm$,

$$\Omega_1^\pm(h) = h - h_\pm,$$

$$\Omega_2^\pm(h) = (h - h_\pm)^2,$$

$$\Omega_3^\pm(h) = \frac{2}{(4 - h^2)^{3/2}} (h_\pm h^3 - 6h^2 + 16 - 2h_\pm^2) - 2\sqrt{4 - h_\pm^2},$$

$$\Omega_4^\pm(h) = -\frac{24}{(4 - h^2)^3} \left((h_\pm^2 - 2)h^2 - 4h_\pm h + 16 - 2h_\pm^2 + \sqrt{4 - h_\pm^2} \sqrt{4 - h^2} (-4 + h_\pm h) \right).$$

Obviously, $\Omega_1^\pm(h)$ and $\Omega_2^\pm(h)$ do not vanish when $h \in U^\pm$. By direct computation, we obtain that $\Omega_4^\pm(h)$ has only one zero $h = h_\pm$ with multiplicity four, which implies that $\Omega_4^\pm(h)$ also does not change sign in each interval. However, it is a little difficult to judge the sign of $\Omega_3^\pm(h)$.

Proposition 11. $(f_0^-(h), f_1^-(h), f_2^-(h), f_3^-(h))$ is an ECT-system on U^- . If $b^2 + c^2 \leq 4$, $(f_0^+(h), f_1^+(h), f_2^+(h), f_3^+(h))$ is an ECT-system on U^+ ; if $b^2 + c^2 > 4$, there is a real number $d \in U^+$ such that $(f_0^+(h), f_1^+(h), f_2^+(h), f_3^+(h))$ is an ECT-system on (h_+, d) .

Proof. It is easy to verify that $\Omega_3^\pm(h_\pm) = 0$ and

$$\Omega_3^{\pm'}(h) = -\frac{12h(h-h_\pm)^2}{(4-h^2)^{5/2}}. \quad (4.5)$$

Thus $\Omega_3^{\pm'}(h)$ has zeros $h = 0$ and $h = h_\pm$ with multiplicity one and two, respectively.

Note that $U^- = (-2, h_-)$ and $h_- < 0$. If $h \in U^-$, by (4.5), we find that $\Omega_3^{-'}(h) > 0$ holds, which implies that $\Omega_3^-(h) < \Omega_3^-(h_-) = 0$. Therefore combining with the discussions above and lemma 10, $(f_0^-(h), f_1^-(h), f_2^-(h), f_3^-(h))$ is an ECT-system on U^- .

We split the proof into two cases if $h \in U^+$, i.e., $h \in (h_+, 2)$.

Case 1. $h_+ \geq 0$, i.e., $b^2 + c^2 \leq 4$.

By (4.5), $\Omega_3^{+'}(h) < 0$ holds on U^+ . It follows that $\Omega_3^+(h) < \Omega_3^+(h_+) = 0$. Thus $\Omega_3^+(h)$ does not vanish on U^+ , which shows that $(f_0^+(h), f_1^+(h), f_2^+(h), f_3^+(h))$ is an ECT-system on U^+ .

Case 2. $h_+ < 0$, i.e., $b^2 + c^2 > 4$.

We get $\Omega_3^{+'}(h) > 0$ for $h \in (h_+, 0)$ and $\Omega_3^{+'}(h) < 0$ for $h \in (0, 2)$ by (4.5). Noting that $\Omega_3^+(h_+) = 0$, $\Omega_3^+(0) = (\sqrt{4-h_+^2}-2)^2/2 > 0$ and $\Omega_3^+(h) \rightarrow -\infty$ as $h \rightarrow 2^-$, we know that there is a simple zero of $\Omega_3^+(h)$ on $(0, 2)$. Thus $(f_0^+(h), f_1^+(h), f_2^+(h), f_3^+(h))$ is not an ECT-system on U^+ by lemma 10. But there is a real number $d \in U^+$ such that $\Omega_3^+(h) > 0$ for $h \in (h_+, d)$. Consequently, $(f_0^+(h), f_1^+(h), f_2^+(h), f_3^+(h))$ is an ECT-system on (h_+, d) .

The proof is finished. \square

Next, let us consider system $X_{2.10}$.

Similarly, the Abelian integral associated to system $X_{2.10}$ has the form:

$$\begin{aligned} M(h) &= \sum_{i+j=0}^3 (-a_{ji} - b_{ij}) I_{ij}(h) \\ &= a_1 + a_2 h + a_3 h^2 + a_4 \sqrt{4-h^2} + a_5 h \sqrt{4-h^2}, \quad h \in (2-b^2, 2), \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} a_2 &= -\frac{4(b_{00} + a_{00})(4-6b^2+b^4)\pi}{(4-b^2)^{3/2}} - \frac{8(b_{10} + a_{01} + b_{30} + a_{03})b\pi}{(4-b^2)^{3/2}} - \frac{2(b_{01} + a_{10})b\pi}{\sqrt{4-b^2}} \\ &\quad + \frac{4(b_{20} + a_{02})b^2\pi}{(4-b^2)^{3/2}} + \frac{2(b_{11} + a_{11})\pi}{\sqrt{4-b^2}} - \frac{(b_{21} + a_{12} - b_{03} - a_{30})b\pi}{\sqrt{4-b^2}}, \\ a_3 &= \frac{4(b_{00} + b_{20} + a_{00} + a_{02})\pi}{(4-b^2)^{3/2}} - \frac{2(b_{10} + a_{01})b\pi}{(4-b^2)^{3/2}} + \frac{(b_{30} + a_{03})b(-6+b^2)\pi}{(4-b^2)^{3/2}}, \\ a_4 &= (b_{21} - b_{03} + a_{12} - a_{30})\pi, \\ a_5 &= (b_{30} + a_{03})\pi, \end{aligned}$$

and

$$a_1 = ma_2 + na_3 + pa_4 + qa_5, \quad (4.7)$$

with

$$m = -2 + b^2, \quad n = -(-2 + b^2)^2, \quad p = -b\sqrt{4 - b^2}, \quad q = b\sqrt{4 - b^2}(-2 + b^2).$$

It follows from

$$\frac{\partial(a_2, a_3, a_4, a_5)}{\partial(a_{11}, a_{01}, a_{12}, a_{03})} = -\frac{4b\pi^4}{(4 - b^2)^2} \neq 0$$

that a_2, a_3, a_4, a_5 are independent.

Denote

$$f_0(h) = h + m, f_1(h) = h^2 + n, f_2(h) = \sqrt{4 - h^2} + p, f_3(h) = h\sqrt{4 - h^2} + q.$$

By (4.6) and (4.7),

$$M(h) = a_2f_0(h) + a_3f_1(h) + a_4f_2(h) + a_5f_3(h). \quad (4.8)$$

Proposition 12. *When $0 < b \leq \sqrt{2}$, $(f_0(h), f_1(h), f_2(h), f_3(h))$ is an ECT-system on $(2 - b^2, 2)$; when $\sqrt{2} < b < 2$, there is a real number $d \in (2 - b^2, 2)$ such that $(f_0(h), f_1(h), f_2(h), f_3(h))$ is an ECT-system on $(2 - b^2, d)$.*

Proof. By direct computations,

$$\begin{aligned} \Omega_1(h) &= h + b^2 - 2, \\ \Omega_2(h) &= (h + b^2 - 2)^2, \\ \Omega_3(h) &= \frac{2}{(4 - h^2)^{3/2}} ((2 - b^2)h^3 - 6h^2 - 2(-4 - 4b^2 + b^4)) - 2b\sqrt{4 - b^2}, \\ \Omega_4(h) &= -\frac{24}{(4 - h^2)^3} ((2 - 4b^2 + b^4)h^2 - 4(2 - b^2)h + 8 + 8b^2 - 2b^4 \\ &\quad + b\sqrt{4 - b^2}\sqrt{4 - h^2}(-4 + (2 - b^2)h)), \end{aligned}$$

where $h \in (2 - b^2, 2)$.

Obviously, $\Omega_1(h)$ and $\Omega_2(h)$ do not vanish on $(2 - b^2, 2)$. It is easy to verify that $h = 2 - b^2$ is a unique zero of $\Omega_4(h)$ with multiplicity four, which implies that $\Omega_4(h) \neq 0$ for $h \in (2 - b^2, 2)$. Therefore we just need to determine the sign of $\Omega_3(h)$. The discussion is similar to the proof of Proposition 11 for $h \in U^+$, thus we omit it. \square

Proof of Theorem 4. It follows from Theorem 1 that $H_{X_{2.9}}(3) \leq 3$. By Proposition 11 and Remark 9, we know that there exist $a_i^\pm, i = 2, 3, 4, 5$, such that the Abelian integral $M^\pm(h)$ has exactly three zeros on U^\pm , respectively. Therefore $H_{X_{2.9}}(3) = 3$.

Similarly, $H_{X_{2.10}}(3) = 3$ by Theorem 2, Proposition 12 and Remark 9.

5 Simultaneous bifurcation and distribution of limit cycles for system $X_{2,9}$

In what follows we will consider the simultaneous bifurcation of limit cycles bifurcating from the two period annuli of system $X_{2,9}$ under cubic polynomial perturbations. Note that $a_4^- = -a_4^+$ and $a_5^- = -a_5^+$. Rewrite (4.3) as

$$M^\pm(h) = a_2^\pm f_0^\pm(h) + a_3^\pm f_1^\pm(h) \pm a_4^+ f_2^\pm(h) \pm a_5^+ f_3^\pm(h), \quad h \in U^\pm. \quad (5.1)$$

Proof of Theorem 5. We study the number of zeros of the two Abelian integrals simultaneously. Note that

$$\frac{\partial(a_2^+, a_2^-, a_3^+, a_3^-, a_4^+, a_5^+)}{\partial(a_{11}, b_{11}, b_{00}, a_{01}, b_{21}, b_{30})} = \frac{128c\pi^6}{(4-b^2)^2(4-c^2)^2} \neq 0.$$

Thus, we can consider $a_2^+, a_2^-, a_3^+, a_3^-, a_4^+, a_5^+$ to be independent.

Firstly, let $a_3^+ = a_3^- = a_4^+ = a_5^+ = 0$ and $a_2^+, a_2^- \neq 0$, then neither $M^+(h)$ has zeros on U^+ nor $M^-(h)$ has zeros on U^- . Hence, the distribution (0,0) is possible.

Secondly, let $a_4^+ = a_5^+ = 0$. By Proposition 11 and Remark 9, we can choose a_2^+, a_3^+ (resp. a_2^-, a_3^-) such that $M^+(h)$ (resp. $M^-(h)$) has none or one zero on U^+ (resp. U^-). Thus the distributions (1,0), (0,1) and (1,1) can be achieved.

Thirdly, take $a_5^+ = 0$. It follows from Proposition 11 and Remark 9 that there exist $\rho_1, \rho_2, \rho_3 (\neq 0)$ and $\sigma_1, \sigma_2, \sigma_3 (\neq 0)$ such that

$$\rho_1 f_0^+(h) + \rho_2 f_1^+(h) + \rho_3 f_2^+(h) \quad \text{and} \quad \sigma_1 f_0^-(h) + \sigma_2 f_1^-(h) - \sigma_3 f_2^-(h)$$

have u, v zeros respectively, with $0 \leq u, v \leq 2$. Then multiplying by ρ_3/σ_3 the second function, it turns out that

$$\sigma_1 \rho_3 / \sigma_3 f_0^-(h) + \sigma_2 \rho_3 / \sigma_3 f_1^-(h) - \rho_3 f_2^-(h)$$

has v zeros. Choose $a_2^+, a_2^-, a_3^+, a_3^-, a_4^+$ such that

$$a_2^+ = \rho_1, \quad a_2^- = \sigma_1 \rho_3 / \sigma_3, \quad a_3^+ = \rho_2, \quad a_3^- = \sigma_2 \rho_3 / \sigma_3, \quad a_4^+ = \rho_3.$$

It follows that the distribution $(u, v), 0 \leq u, v \leq 2$ is possible.

Finally, suppose that $a_5^+ \neq 0$. It is easy to obtain that

$$M^{+(3)}(h) = -\frac{12(a_4^+ h + 4a_5^+)}{(4-h^2)^{5/2}} \quad \text{and} \quad M^{-{(3)}}(h) = \frac{12(a_4^+ h + 4a_5^+)}{(4-h^2)^{5/2}}.$$

If $a_4^+ = 0$, then $M^+(h)$ and $M^-(h)$ have at most two zeros on U^+ and U^- , respectively. Thus, in order to get more limit cycles, $a_4^+ \neq 0$. We find that $M^{+(3)}(h)$ and $M^{-(3)}(h)$ have the same unique zero $h_0 = -4a_5^+/a_4^+$, which implies that $M^{+(2)}(h)$

has at most two (resp. one) zero(s) on U^+ and $M^{-(2)}(h)$ has at most one (resp. two) zero(s) on U^- if $h_0 \in U^+$ (resp. $h_0 \in U^-$), otherwise, both $M^{+(2)}(h)$ and $M^{-(2)}(h)$ have at most one zero on U^+ and U^- , respectively. Thus, $M^+(h)$ has at most three (resp. two) zeros on U^+ and $M^-(h)$ has at most two (resp. three) zeros on U^- , which means the distribution (3,3) is impossible. To show (3,2) is achievable, we give the following asymptotic expansions of $M^\pm(h)$ at $h = h_\pm$, respectively:

$$M^\pm(h) = s_1^\pm(h - h_\pm) + s_2^\pm(h - h_\pm)^2 + s_3^\pm(h - h_\pm)^3 + s_4^\pm(h - h_\pm)^4 + \cdots, \quad (5.2)$$

where

$$\begin{aligned} s_1^\pm &= a_2^\pm + 2a_3^\pm h_\pm \mp \frac{a_4^+ h_\pm + 2a_5^+(h_\pm^2 - 2)}{(4 - h_\pm^2)^{1/2}}, & s_2^\pm &= a_3^\pm \mp \frac{2a_4^+ - a_5^+ h_\pm (h_\pm^2 - 6)}{(4 - h_\pm^2)^{3/2}}, \\ s_3^\pm &= \mp \frac{2(a_4^+ h_\pm + 4a_5^+)}{(4 - h_\pm^2)^{5/2}}, & s_4^\pm &= \mp \frac{2(a_4^+ + 5a_5^+ h_\pm + a_4^+ h_\pm^2)}{(4 - h_\pm^2)^{7/2}}. \end{aligned}$$

Since

$$\frac{\partial(s_1^+, s_1^-, s_2^+, s_2^-, s_3^+, s_4^+)}{\partial(a_2^+, a_2^-, a_3^+, a_3^-, a_4^+, a_5^+)} = -\frac{4}{(4 - h_+^2)^5} \neq 0,$$

we consider $s_1^+, s_1^-, s_2^+, s_2^-, s_3^+, s_4^+$ as the new independent parameters. Denote

$$M^+(h) = M^+(h, s_1^+, s_2^+, s_3^+, s_4^+) \quad \text{and} \quad M^-(h) = M^-(h, s_1^-, s_2^-, s_3^-, s_4^-).$$

Without loss of generality, suppose $s_4^+ > 0$. To get more zeros of $M^+(h)$, we choose s_i^+ and $h_i \in U^+$, $i = 4, 3, 2, 1$, such that

$$\begin{aligned} M^+(h_4, 0, 0, 0, s_4^+) &> 0, & M^+(h_3, 0, 0, s_3^+, s_4^+) &< 0, \\ M^+(h_2, 0, s_2^+, s_3^+, s_4^+) &> 0, & M^+(h_1, s_1^+, s_2^+, s_3^+, s_4^+) &< 0, \end{aligned} \quad (5.3)$$

and $0 < |s_1^+| \ll |s_2^+| \ll |s_3^+| \ll |s_4^+|$, $h_+ < h_1 < h_2 < h_3 < h_4 < 2$. It is easy to show that $M^+(h)$ has three zeros which tend to h_+ . Once s_3^+, s_4^+ are chosen, the sign of s_3^- is determined. For example, from the analysis above, we know that $s_4^+ > 0$ and $s_3^+ < 0$ by (5.3). The result that $M^+(h)$ has three zeros on U^+ implies $h_0 = -4a_5^+/a_4^+ \in U^+$. Thus $a_4^+ < 0$ and $s_3^- > 0$. There exists $h_5 \in U^-$ such that $M^-(h_5, 0, 0, s_3^-, s_4^-) < 0$. Take $s_i^-, i = 2, 1$ and $h_i \in U^-, i = 6, 7$, such that

$$M^-(h_6, 0, s_2^-, s_3^-, s_4^-) > 0, \quad M^-(h_7, s_1^-, s_2^-, s_3^-, s_4^-) < 0,$$

and $0 < |s_1^-| \ll |s_2^-| \ll \min\{|s_3^-|, |s_4^-|\}$, $-2 < h_5 < h_6 < h_7 < h_-$. It follows that $M^-(h)$ has at least two zeros on U^- , which tend to h_- . Thus, (3,2) is realizable. Similarly, other configurations of limit cycles (u, v) with $0 \leq u, v \leq 3, u + v \leq 5$ can be realized in this way.

This completes the proof.

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